J. Fluid Mech. (1971), *vol.* 46, *part* 4, *pp.* 727–736
Printed in Great Britain

Boundary-layer separation at a free streamline Part 2. Numerical results

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(Received 8 September 1970)

An asymptotic solution of the boundary-layer equations, valid just upstream of a free streamline attached to the sharp trailing edge of a body, is compared with a numerical solution for the boundary-layer flow on a finite flat plate set perpendicular to a uniform stream. An arbitrary multiplicative constant in the asymptotic expansion, arising from an eigenfunction, is evaluated by requiring the skin friction to agree with a numerical value close to the free streamline. Using this value, the velocity profiles, computed from the asymptotic expansion, are in excellent agreement with the numerical solution.

1. Introduction

In a recent paper, Ackerberg (1970) [hereafter referred to as I] proposed an asymptotic theory to describe the nature of the boundary-layer flow just upstream of a free streamline which is attached to the sharp trailing edge of a body. These problems are unusual because the pressure distribution in the potential flow exhibits a singularity which appears in the boundary-layer equations as a forcing term. It is of interest to know how this singularity influences the boundary-layer motion and the skin friction near the edge. The work presented here compares a numerical solution of the boundary-layer equations for the flow over a finite flat plate set perpendicular to a uniform stream, shown in figure 1, with the asymptotic theory. Outside the boundary layer, a potential motion of the Kirchhoff–Rayleigh type with free streamlines attached at the sharp edges, is assumed.

An explicit finite difference technique was used to solve the boundary-layer equations in Mises variables. This method was introduced by Mitchell & Thomson (1958), and has been used by Ackerberg (1968) to study the development of the boundary-layer motion in a thin film flowing along a vertical plate. The new features which enter our calculation are as follows: (1) The separation at the free streamline occurs with an extremely *favourable* pressure gradient which is singular at the edge; thus, the small step size of the explicit technique is useful in suppressing the truncation errors. (2) An asymptotic formula derived by Brown & Stewartson (1965) is incorporated into the calculation of the velocity at the outermost point of the boundary layer. (3) A spurious singularity enters the numerical solution as a result of truncating a series expansion, valid near the wall, which is used to calculate the skin friction.

In § 2, the problem is formulated mathematically and cast into finite difference form. The asymptotic results in I are extended and summarized in § 3, and in § 4 a comparison is made between the numerical results and the asymptotic theory. The results and conclusions are discussed in § 5. It was found that accurate numerical results can be obtained close to the free streamline in spite of the large truncation errors, and excellent agreement was obtained with the asymptotic theory.

2. Mathematical and numerical formulations

Mathematical formulation

A co-ordinate system is introduced with the $\bar{x}$ axis coincident with the plate and directed toward a free streamline, and the $\bar{y}$ axis perpendicular to it and directed

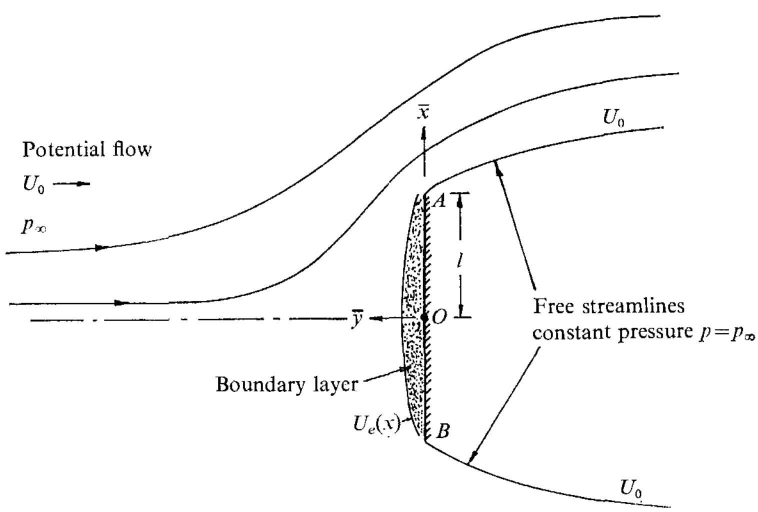

FIGURE 1. Geometry of flat plate set perpendicular to a uniform stream.

into the fluid (see figure 1). The origin is at the midpoint of the plate. Denoting dimensional variables by bars, the following non-dimensionalization is introduced:

$$\left.\begin{aligned} x = \bar{x}/L, \quad Y = \bar{y}\,Re^{\frac{1}{2}}/L, \quad u = \bar{u}/U_0, \quad v = \bar{v}\,Re^{\frac{1}{2}}/U_0, \\ \psi(x, Y) = \bar{\psi}(\bar{x}, \bar{y})\,Re^{\frac{1}{2}}/U_0 L. \end{aligned}\right\} \tag{2.1}$$

Here u, v are the velocity components in the x and y directions, and ψ is the stream function with $u = \partial\psi/\partial Y$ and $v = -\partial\psi/\partial x$. The length $L = 4(\pi+4)^{-1}l$, l being the half-breadth of the plate, U_0 is the fluid speed along a free streamline, $Re = \rho U_0 L/\mu$ is the Reynolds number, and ρ and μ are the constant fluid density and viscosity. In place of the variable x, it is convenient to introduce the new independent variable

$$s = U_e(x), \tag{2.2}$$

where $U_e(x)$ is the non-dimensional x component of velocity just outside the boundary layer determined from the potential flow solution. In the problem considered here, $(0 \leqslant s \leqslant 1)$, and in most applications $U_e(x)$ increases monotonically as the free streamline is approached so the transformation will be bi-uniform. As dependent variable we use

$$p(s, \psi) = u^2(s, \psi)/s^2 \quad (0 \leqslant p < 1). \tag{2.3}$$

Substituting (2.2) and (2.3) into the boundary-layer equation written in Mises co-ordinates (see Goldstein 1938, p. 126) we obtain

$$\partial p/\partial s = (2/s)(1-p) + [s/U_e'(s)]p^{\frac{1}{2}}\,\partial^2 p/\partial \psi^2 \quad (0 \leqslant s < 1,\, 0 \leqslant \psi < \infty), \tag{2.4}$$

where

$$U_e'(s) = dU_e(x)/dx = \tfrac{1}{4}(1+s^2)^3/(1-s^2). \tag{2.5}$$

This last equation follows from the relationship

$$x(s) = s(s^2+3)/(1+s^2)^2 + \tfrac{1}{2}\sin^{-1}[2s/(1+s^2)], \tag{2.6}$$

which may be derived from the potential solution given in Lamb (1945, p. 99). Here the value of the inverse sine is in the range $(0, \frac{1}{2}\pi)$. A singularity in U_e' is apparent when $s \to 1$; in terms of $\bar{x}$, the singularity is of $O[(l-\bar{x})^{-\frac{1}{2}}]$ for $\bar{x} \to l$.

The usual boundary conditions at the wall and at the edge of the boundary layer require

$$p(s, 0) = 0 \quad (0 \leqslant s < 1), \tag{2.7}$$

and

$$p(s, \psi) \to 1 \quad \text{for} \quad \psi \to \infty \quad (0 \leqslant s < 1). \tag{2.8}$$

We also require an initial condition to specify that the boundary-layer flow starts at the stagnation point O with a velocity profile given by the Falkner–Skan similarity solution with parameter $\beta = 1$. This solution may be obtained in our co-ordinate system by putting $U_e' = \frac{1}{4}$ (i.e. the limiting value for $s \to 0$) and assuming

$$p(s, \psi) = F(\zeta), \quad \text{where} \quad \zeta = \psi/s. \tag{2.9}$$

Substituting (2.9) in (2.4) leads to a non-linear ordinary differential equation for $F(\zeta)$ which should be solved with some care due to a singular point at $\zeta = 0$. It might be thought desirable to obtain a number of terms in an asymptotic expansion near $s = 0$; however, experience has shown that the flow near the free streamline is quite insensitive to the initial condition, and putting $F(\zeta) \equiv 1$ at a small value of s (which we took as $s = 0{\cdot}1$) does not affect the final results.

Finite difference equations

Introduce the finite difference notation

$$p_{j,k} = p(s_j, \psi_k), \quad \partial p/\partial s = (p_{j+1,k} - p_{j,k})/\Delta s + O(\Delta s), \tag{2.10}$$

and

$$\partial^2 p/\partial \psi^2 = (p_{j,k+1} - 2p_{j,k} + p_{j,k-1})/(\Delta\psi)^2 + O[(\Delta\psi)^2], \tag{2.11}$$

where Δs and $\Delta\psi$ denote constant mesh widths in the s and ψ directions. Let the plate be specified by $k = 1$, the point at the outer edge of the boundary layer by $k = M$ and the initial station, $s_1 = 0{\cdot}1$, by $j = 1$. When (2.10) and

(2.11) are substituted in (2.4), using (2.5), we obtain the explicit finite difference formula

$$p_{j+1,k} = p_{j,k} + (2\Delta s/s_j)(1-p_{j,k}) + 4hs_j(1-s_j^2) \times (1+s_j^2)^{-3} p_{j,k}^{\frac{1}{2}} (p_{j,k+1} - 2p_{j,k} + p_{j,k-1}), \quad (2.12)$$

with $h = \Delta s/(\Delta\psi)^2$. At any internal point $3 \leqslant k \leqslant M-1$, (2.12) will be used to compute $p_{j+1,k}$ from the values $p_{j,k}$; the other points require special consideration.

The boundary condition at the wall (2.7) and the initial condition (2.9) are satisfied if

$$p_{j,1} = 0 \quad \text{for} \quad j \geqslant 1, \quad (2.13)$$

and

$$p_{1,k} = F(\psi_k/s_1) \quad \text{for} \quad 1 \leqslant k \leqslant M. \quad (2.14)$$

We cannot choose $s_1 \equiv 0$ because (2.4) is singular at $s = 0$.

The point at the edge of the boundary layer, $k = M$, is computed using an asymptotic formula derived by Brown & Stewartson (1965), which we have expressed in Mises co-ordinates, i.e.

$$p^{\frac{1}{2}} \to 1 + \alpha_0 \psi^n \exp[-\psi^2/G(s) + \alpha_1\psi] \quad \text{for} \quad \psi \to \infty, \quad (2.15)$$

where α_0, α_1 and n are constants, for a fixed s, which are determined by fitting (2.15) to the three values $p_{j,k}$ for $k = M-3$, $M-2$, and $M-1$. Here

$$G(s) = 4\int_0^{x(s)} U_e(x)\,dx = 8s^2(1+s^2)^{-2}. \quad (2.16)$$

Once values for α_0, α_1 and n are found, (2.15) is used to compute $p_{j,M}$. This value is assigned to the point at the edge of the boundary layer and a test is made to determine if $p_{j,M}^{\frac{1}{2}} (= u_{j,M}/U_e)$ is less than a preassigned tolerance, say 0·9999. If it is, a new point is added at the edge of the boundary layer by putting $p_{j,M+1} \equiv 1$.† It is also possible to delete points at the edge if $p_{j,M-1}^{\frac{1}{2}}$ is greater than the tolerance.

Special care is required for $p_{j,2}$. This can be seen from the series expansion of s^2p^2 $(= u^2)$ near the wall,

$$s^2p^2 = a^2\psi - (8/3a)(sU_e')\,\psi^{\frac{3}{2}} - (4/3a^4)(sU_e')^2\psi^2 - c\psi^{\frac{5}{2}} \quad \text{for} \quad \psi \to 0. \quad (2.17)$$

Here the skin friction, $\tau_w = u(\partial u/\partial\psi)|_{\psi=0} = \frac{1}{2}a^2$, and c is a constant chosen to make the truncated series accurate. It is apparent that $\partial^2 p/\partial\psi^2$ is infinite at $\psi = 0$ and large truncation errors may occur. To avoid them, the following procedure was adopted. At a given station s_j, (2.17) is applied to each of the values $p_{j,2}$ and $p_{j,3}$ with $\psi = \Delta\psi$ and $\psi = 2\Delta\psi$, respectively. The constant c is then eliminated between the two equations to yield

$$A = (2^{\frac{5}{2}}p_{j,2}^2 - p_{j,3}^2)\,s_j^2[(2^{\frac{5}{2}}-2)\,\Delta\psi]^{-1} + (2^{\frac{7}{2}}/3)\,sU_e'(\Delta\psi)^{\frac{1}{2}}[(2^{\frac{3}{2}}-1)\,A^{\frac{1}{2}}]^{-1} + \tfrac{8}{3}(2^{\frac{1}{2}}-1)(sU_e')^2\Delta\psi[(2^{\frac{3}{2}}-1)\,A^2]^{-1}, \quad (2.18)$$

where $A = a^2$. Using the best value of A at the previous step s_{j-1}, we solve (2.18) by a fixed point iteration, i.e. $A_{n+1} = H(A_n)$, where $H(A)$ is the function on the

† Alternatively, (2.15) could be used to compute the value $p_{j,M+1}$ and other values (if necessary) until the tolerance is exceeded.

right-hand side of (2.18). After three iterations, 6 or 7 significant figures were obtained and the value $\frac{1}{2}A$ gave a running account of the skin friction. The constant c is then determined by substituting A into one of the original equations. The value $p_{j+1,2}$ is found using the forward difference

$$\begin{aligned} p_{j+1,2} &= p_{j,2} + (\partial p/\partial s)_{j,2}\Delta s + O[(\Delta s)^2] \\ &= p_{j,2} + (2/s_j)(1-p_{j,2})\,\Delta s + 4s_j(1-s_j^2)(1+s_j^2)^{-3}\,\Delta s[p^{\frac{1}{2}}\,\partial^2 p/\partial\psi^2]_{j,2}, \end{aligned} \quad (2.19)$$

where (2.4) has been used to obtain the last line. The quantity $[p^{\frac{1}{2}}\,\partial^2 p/\partial\psi^2]_{j,2}$ can be evaluated analytically using the series expansion (2.17). After some algebra we find

$$[p^{\frac{1}{2}}\,\partial^2 p/\partial\psi^2]_{j,2} = -s_j^{-3}\{2s_j U_e' + [(15ac/4) - (20/3a^6)(s_j U_e')^3]\,\Delta\psi\}, \quad (2.20)$$

where U_e', a, and c are to be evaluated at s_j as the suffix on the left-hand term implies. Substituting (2.20) in (2.19) yields $p_{j+1,2}$ with an error of $O[\Delta s\Delta\psi^{\frac{3}{2}}, (\Delta s)^2]$; however, since values of $p_{j,k}$ for $k \geqslant 3$ will be known only with an accuracy of $O[(\Delta\psi)^2]$, the error in $p_{j,2}$ is likely to be the same.

To prevent the growth of truncation errors during the calculation, a stability criterion based on the linear heat conduction equation was used. This requires

$$\Delta s/(\Delta\psi)^2 < \min_{(0\leqslant s\leqslant 1)} U_e'/(2sp^{\frac{1}{2}}) \geqslant \min_{(0\leqslant s\leqslant 1)} (U_e'/2s) = 0{\cdot}57496.... \quad (2.21)$$

If a fixed Δs was used throughout the interval, (2.21) must be satisfied for stability. However, since the smallest value of $(U_e'/2s)$ occurs for $s_0 = 0{\cdot}3626...$, larger step sizes may be taken on either side of s_0.

3. Summary and extension of asymptotic results

An asymptotic solution valid near the wall, for $s \to 1$, has been found with the form

$$\psi(t, Y) = 2^{\frac{1}{2}}kt^{\frac{3}{4}}\{F_0(\eta) + Ct^{\gamma}F_{\gamma}(\eta) + C^2t^{2\gamma}F_{2\gamma}(\eta) + C^3t^{3\gamma}F_{3\gamma}(\eta) + tF_1(\eta) + o(t)\}, \quad (3.1)$$

where the similarity variable

$$\eta = 2^{\frac{1}{2}}kY/t^{\frac{1}{4}}, \quad t = 1 - U_e(x) \quad (t \geqslant 0), \quad (3.2)$$

k is a constant which may be found from the potential flow solution,† C is a constant which depends on the boundary-layer flow upstream, and the functions $F_\alpha(\eta)$ are solutions of the following ordinary differential equations:

$$F_0''' - \tfrac{5}{4}F_0F_0'' + \tfrac{1}{2}F_0'^2 + 1 = 0, \quad (3.3)$$

$$F_{n\gamma}''' - \tfrac{5}{4}F_0F_{n\gamma}'' + (n\gamma+1)F_0'F_{n\gamma}' - [n\gamma + \tfrac{5}{4}]F_0''F_{n\gamma} = G_{n\gamma} \quad (n = 1, 2, 3), \quad (3.4)$$

$$F_1''' - \tfrac{5}{4}F_0F_1'' + 2F_0'F_1' - \tfrac{9}{4}F_0''F_1 = 1 - \tfrac{5}{2}F_0''', \quad (3.5)$$

subject to the boundary conditions

$$F_\alpha(0) = 0 = F_\alpha'(0) \quad (\alpha \geqslant 0), \quad (3.6)$$

† For the flow in figure 1, $k = 2^{-\frac{1}{2}}$.

and

$$F_\alpha(\eta) \text{ must not contain any exponentially large terms for } \eta \to \infty. \tag{3.7}$$

Here, $\gamma = 0{\cdot}3157\ldots$ is an eigenvalue which was computed in I, and

$$\left.\begin{aligned} G_\gamma \equiv 0, \quad G_{2\gamma} &= (\gamma+\tfrac{5}{4})F_\gamma F''_\gamma - (\gamma+\tfrac{1}{2})F'^2_\gamma, \\ G_{3\gamma} &= (\gamma+\tfrac{5}{4})\,F_\gamma F''_{2\gamma} + (2\gamma+\tfrac{5}{4})\,F''_\gamma F_{2\gamma} - (3\gamma+1)\,F'_\gamma F'_{2\gamma}. \end{aligned}\right\} \tag{3.8}$$

The function F_0 was calculated in I. The functions $F_{n\gamma}$ $(n = 1, 2, 3)$ and F_1 have been obtained numerically using a method described in the appendix and are shown in figures 2 and 3.

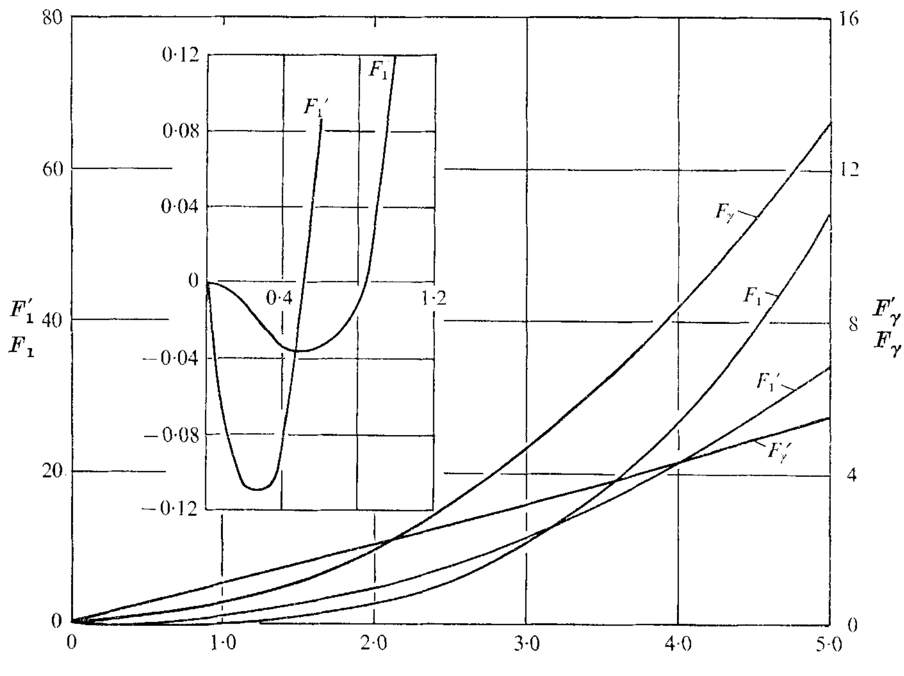

FIGURE 2. $F_\gamma(\eta)$, $F'_\gamma(\eta)$ and $F_1(\eta)$, $F'_1(\eta)$ versus η.
$F''_1(0) = 0{\cdot}88547\ldots$, $F''_\gamma(0) = 1{\cdot}0$.

The streamwise velocity component and the skin friction are

$$u(t, Y)/U_e = 2k^2t^{\frac{1}{2}}(1-t)^{-1}\,[F'_0(\eta) + Ct^\gamma F'_\gamma(\eta) + C^2t^{2\gamma}F'_{2\gamma}(\eta) + C^3t^{3\gamma}F'_{3\gamma}(\eta) + tF'_1(\eta) + o(t)], \tag{3.9}$$

and

$$\tau_w = \partial u/\partial Y|_{Y=0} = (2^{\frac{1}{2}}k)^3\, t^{-\frac{1}{4}}[a_0 + a_\gamma Ct^\gamma + a_{2\gamma}C^2t^{2\gamma} + a_{3\gamma}C^3t^{3\gamma} + a_1 t + o(t)]. \tag{3.10}$$

The constants $a_\alpha = F''_\alpha(0)$ were calculated numerically and have the values

$$a_0 = 3{\cdot}014\ldots, \quad a_\gamma \equiv 1, \quad a_{2\gamma} = -2{\cdot}700\ldots\times 10^{-2}, \quad a_{3\gamma} = 4{\cdot}145\ldots\times 10^{-4}$$

and

$$a_1 = -0{\cdot}8854\ldots.$$

Equation (3.10) indicates that, as the free streamline is approached, the skin friction is singular to first order and proportional to $(l-\bar{x})^{-\frac{1}{3}}$. The coefficient of this term depends only on the potential flow solution, via k, and the boundary-layer flow upstream influences the skin friction, starting at the second-order term, through the constant C. This constant arises in the asymptotic theory as the arbitrary multiple of the eigensolution $F_\gamma(\eta)$. In the following section the numerical results will be compared with the theory presented here.

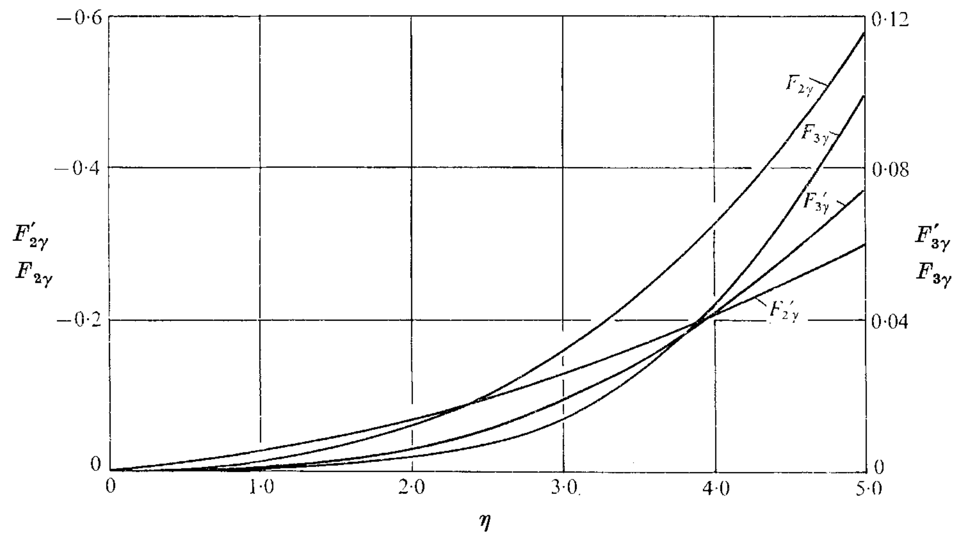

FIGURE 3. $F_{2\gamma}(\eta)$, $F'_{2\gamma}(\eta)$ and $F_{3\gamma}(\eta)$, $F'_{3\gamma}(\eta)$ versus η. $F''_{2\gamma}(0) = -2{\cdot}7003\ldots \times 10^{-2}$, $F''_{3\gamma}(0) = 4{\cdot}1458\ldots \times 10^{-4}$.

4. Comparison of numerical results with the asymptotic theory

The finite difference equations in §2 were programmed for an IBM 360–50 in double precision, and a CDC-6600 in single precision. Four runs were made with $\Delta\psi = 0{\cdot}04, 0{\cdot}02, 0{\cdot}01, 0{\cdot}005$ (hereafter these runs are referred to as 1, 2, 3, 4 respectively). Only run 4 was made on the CDC-6600 with $\Delta s = 1{\cdot}40625 \times 10^{-5}$. At the start of this calculation at $s_1 = 0{\cdot}1$, 137 points of the stagnation point profile were taken inside the boundary layer. The running time, after 64,000 steps, was approximately 26 minutes, and the final profile had 509 points. To satisfy the stability condition (2.21), Δs was reduced by a factor of $\frac{1}{4}$ as $\Delta\psi$ decreased, and this reduction allowed some estimate to be made of the truncation errors. It was found that for $s > 0{\cdot}955$ the results could not be considered reliable due to a singularity inherent in the solution of (2.18) when $s \to 1$. This will be discussed later.

Some velocity profiles from run 4 are shown in figure 4. At least three significant figures were obtained in this calculation for $s \leqslant 0{\cdot}955$. The skin frictions near the free streamline, for all runs, are displayed in figure 5. Values of τ_w for $s > 0{\cdot}955$ have been included for a comparison with the asymptotic theory.

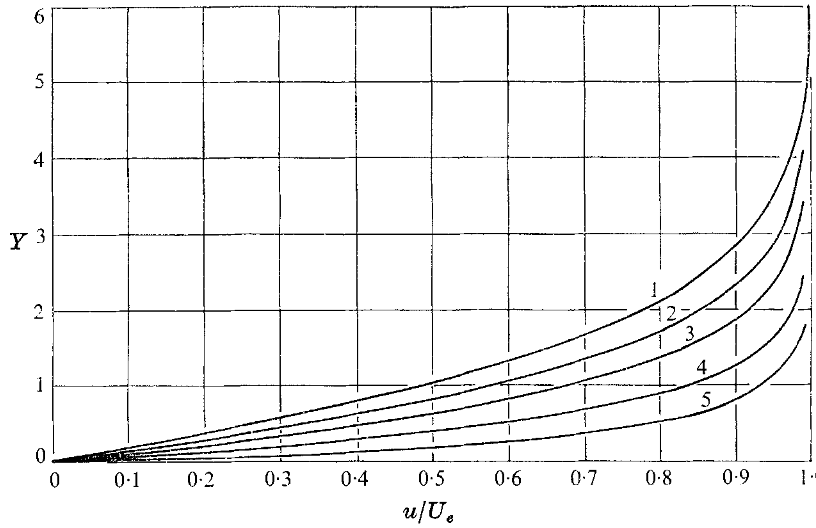

FIGURE 4. Velocity profiles from run 4 at various values of s. Curve: 1, $s = 0{\cdot}100$, $x = 0{\cdot}221$; 2, $s = 0{\cdot}370$, $x = 0{\cdot}701$; 3, $s = 0{\cdot}505$, $x = 0{\cdot}846$; 4, $s = 0{\cdot}775$, $x = 0{\cdot}979$; 5, $s = 0{\cdot}955$, $x = 0{\cdot}999$.

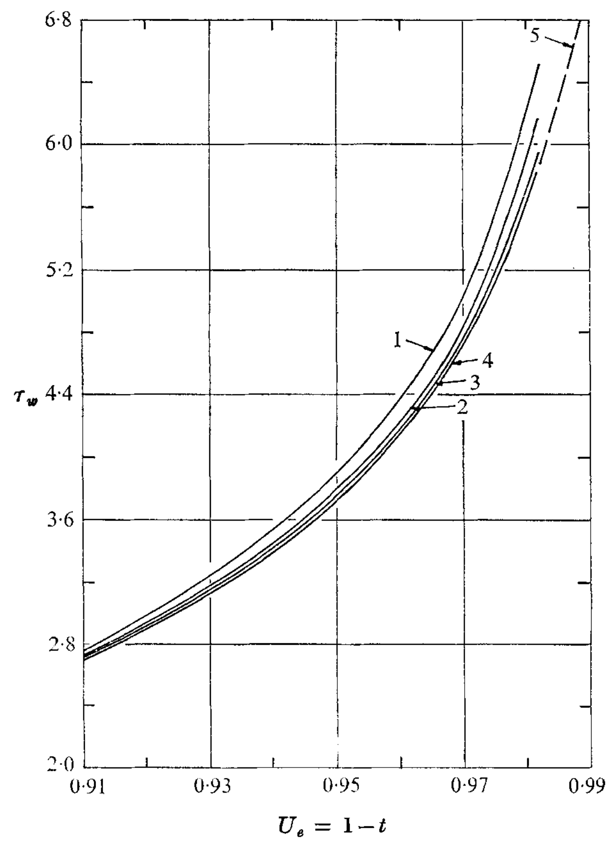

FIGURE 5. Comparison of skin frictions from the finite difference calculations with the asymptotic theoretical result. Curve 5, theoretical result using (3.10) with the value $C = -2{\cdot}989$ obtained by requiring (3.10) to agree with the numerical value from run 4 at $s = 0{\cdot}955$. Numerical results: curve 4, $\Delta\psi = 0{\cdot}005$; curve 3, $\Delta\psi = 0{\cdot}01$; curve 2, $\Delta\psi = 0{\cdot}02$; curve 1, $\Delta\psi = 0{\cdot}04$.

To compare the numerical results with the asymptotic theory, the constant C in (3.1) must be evaluated. C was calculated by requiring that the skin friction, computed from (3.10), agree with a numerical value from run 4 for a small value of t. The solution of the resulting cubic equation in C, which had one real root, gave the desired value. The variation of C with t, i.e. various positions where C was calculated, is shown in table 1. Note the small variation of C for

$\bar{x}/l$	t	C	τ_w(run 4)	τ_w^p
0·9936	0·135	−2·944	2·0827	2·0398
0·9974	0·090	−2·974	2·7215	2·7076
0·9994	0·045	−2·989	3·9446	3·9446
0·9996	0·036	−2·986	4·3775	4·3751

TABLE 1. Variation of C with t. C is determined by requiring (3.10) to agree with a numerical value of τ_w from run 4

the large variation of τ_w. Adopting the value of C corresponding to $t = 0{\cdot}045$, a skin friction curve was predicted from (3.10) and is shown in figure 5 as the dashed curve. In the last column of table 1 some of the predicted values, τ_w^p, are given to make a more detailed comparison.

Velocity profiles were computed from the asymptotic theory for $s = 0{\cdot}865$, 0·9325, 0·955, and are displayed in figure 6 as the dashed curves. The agreement with the numerical solution (solid lines) is quite good in view of the fact that the asymptotic expansion (3.1) is not uniformly valid for large Y as discussed in I.

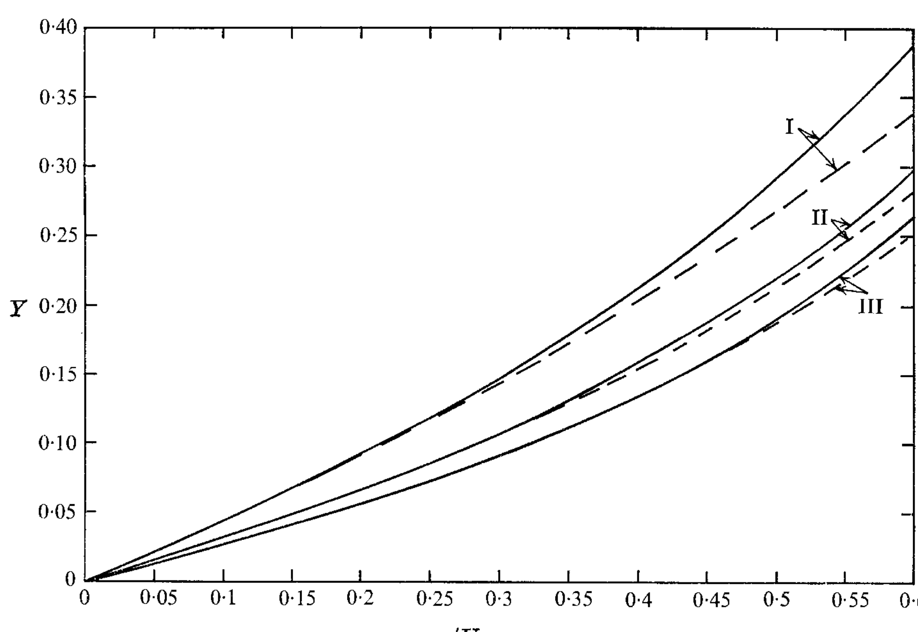

FIGURE 6. I, velocity profiles at $s = 0{\cdot}865$ ($\bar{x}/l = 0{\cdot}9936$), $C = -2{\cdot}9439$; II, velocity profiles at $s = 0{\cdot}9325$ ($\bar{x}/l = 0{\cdot}9986$), $C = -2{\cdot}9845$; III, velocity profiles at $s = 0{\cdot}955$ ($\bar{x}/l = 0{\cdot}9994$), $C = -2{\cdot}9889$; ————, run 4 numerical; – – – – – – – –, computed from the asymptotic solution (3.9).

5. Results and conclusions

The good agreement between the numerical results and the asymptotic theory provides convincing evidence that the boundary-layer flow near a free streamline is well understood. However, the numerical method had these drawbacks:

1. The explicit technique required a very small step size for our most accurate run although a compensating factor is the suppression of the truncation errors that arise from large s derivatives near the free streamline. Recent results, using an implicit technique, indicate that accurate solutions can be obtained near $s = 0{\cdot}955$ with a much larger step size of $\Delta s = 0{\cdot}001$.

2. The equation for the fixed point iteration (2.18) has a singular solution of the form

$$A \propto (\Delta\psi)^{\frac{1}{3}}(1-s)^{-\frac{2}{3}} \quad \text{for} \quad s \to 1. \tag{5.1}$$

This is due to the coefficients which contain the singular term

$$U_e' \propto (1-s)^{-1} \quad \text{for} \quad s \to 1. \tag{5.2}$$

The solution (5.1) was apparent in the numerical results very near the free streamline, and it is not known how this undesirable feature can be eliminated if Mises variables are used. Current calculations using the variables (u, v) and (s, Y) avoid this difficulty, and it is possible to approach the free streamline more closely.

Some care is required in using any numerical method near the free streamline. This can be seen from the form of the streamwise velocity profile at the separation point derived in I; it is

$$U_s(Y) = b_0 Y^{\frac{2}{3}} + b_1 Y^{1{\cdot}087\ldots} + b_2 Y^{1{\cdot}508\ldots} + b_3 Y^{1{\cdot}929\ldots} + b_4 Y^2 + \ldots \quad \text{for} \quad s \equiv 1, \tag{5.3}$$

where the b_i's are constants. At any point upstream, say $s = 1-\epsilon$ for $\epsilon > 0$, the velocity profile is analytic and can be represented by a power series in Y. The transition from an analytic profile to (5.3) will undoubtedly lead to large truncation errors.

The author would like to thank Prof. K. Stewartson for his confidence in the finite difference calculations and for the numerous arguments that led to this comparison of the numerical and asymptotic results. This work was supported by the U.S. Army Research Office-Durham, under Contract No. DA-31-124-ARO-D-444. An abstract of this paper was presented at the Second International Conference on Numerical Methods in Fluid Dynamics at the University of California, Berkeley (1970).

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